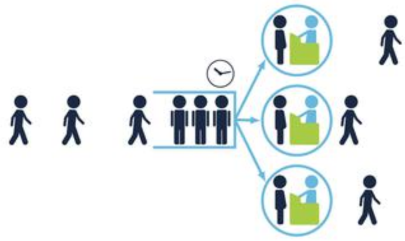

Queuing Theory: from Markov Chains to Multi–Server Systems

Week 5 - Multi-server systems
Lesson 5 - Performance metrics calculation details

1 Notations

C	Number of servers
λ	Arrival rate
μ	Service rate
$\rho = \frac{\lambda}{\mu}$	Offered load
N_w	Mean number of waiting clients
N	Mean number of clients in the system (waiting + service)
R	Mean sojourn time
R_w	Mean waiting time
$E_C(\rho, C)$	Erlang C formula

2 Mean number of clients in the system

Let N_w be the mean number of clients that are waiting in the system.

We can compute N_w as $N_w = \sum_{i=C+1}^{\infty} (i - C)\pi_i$. From the course we know that:

$$\pi_j = \pi_0 \frac{\rho^C}{C!} (\frac{\rho}{C})^{j-C}, \forall j > C, \quad (1)$$

where

$$\pi_0 = \frac{1}{\sum_{j=0}^{C} \frac{\rho^j}{j!} + \frac{\rho^C}{C!} \frac{\rho}{C - \rho}} \quad (2)$$

$$= \frac{E_C(\rho, C)}{\frac{\rho^C}{C!} \frac{C}{C-\rho}}. \quad (3)$$

We can now proceed to the calculation of N_w as follows:

$$
\begin{aligned}
N_w &= \sum_{i=C+1}^{\infty} (i-C)\pi_i && (4)\\
&= \sum_{i=C+1}^{\infty} (i-C)\pi_0 \frac{\rho^C}{C!}\left(\frac{\rho}{C}\right)^{i-C} && (5)\\
&= \frac{\rho^C}{C!}\pi_0 \sum_{j=1}^{\infty}\left(\frac{\rho}{C}\right)^{j} j && (6)\\
&= \frac{\rho^C}{C!}\pi_0 \frac{\frac{\rho}{C}}{(1-\rho/C)^2} = E_C(\rho, C)\frac{\rho}{C-\rho}. && (7)
\end{aligned}
$$

Applying Little's law twice, to the system and to the waiting queue, we have the following two relationships:

$$N = \lambda R \tag{8}$$

and

$$N_w = \lambda R_w. \tag{9}$$

Remembering that average sojourn time is equal to average waiting time plus mean service time we get:

$$R = R_w + \frac{1}{\mu}. \tag{10}$$

Finally, from (8), (9) and (10) we can deduce that $N = N_w + \rho$ which concludes the proof.

3 Mean sojourn time

Mean waiting time can be directly deduced using Little's law, as in (9). We get then that

$$R = \frac{N}{\lambda} = \frac{E_C(\rho, C)}{\mu(C-\rho)} + \frac{1}{\mu}. \tag{11}$$