# Mathematical Tools: Probability Theory, Algebra, ...

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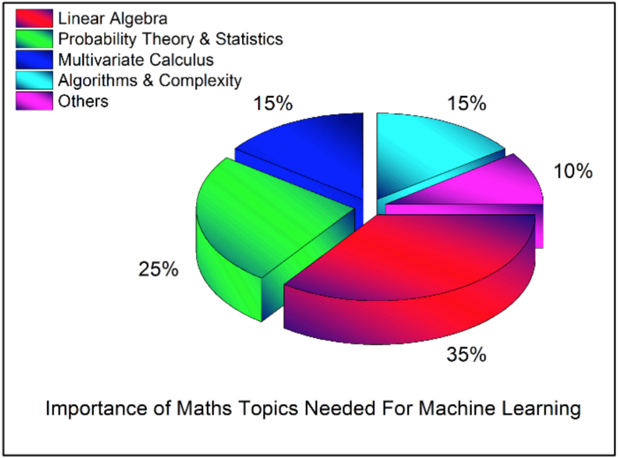
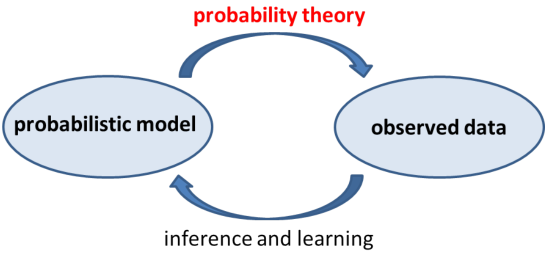



(by Wale Akinfaderin, `https://tinyurl.com/y6hc9sh8`)

# Part I: Probability Theory

# Probability theory




- Probability theory has its roots in games of chance

- Great names of science: Bayes, Bernoulli(s), Boltzman, Cardano, Cauchy, Fermat, Huygens, Kolmogorov, Laplace, Pascal, Poisson, ...
- Tool to handle uncertainty, information, knowledge, observations, ...
- ...thus also learning, decision making, inference, science,...

# Still important today, in the Deep Learning age?

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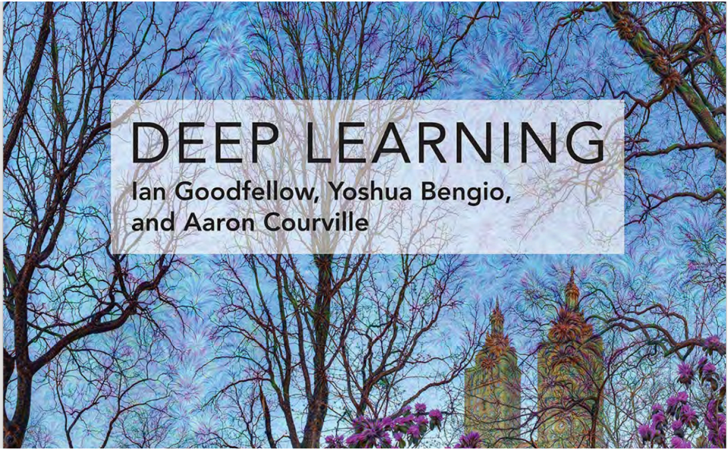
What book is this from?

# Do we still need this?



# What is probability?

Example: $\mathbb{P}(\text{randomly drawn card is } \clubsuit) = 13/52$.

Example: $\mathbb{P}(\text{getting 1 in throwing a fair die}) = 1/6$.

- Classical definition: $\mathbb{P}(A) = \dfrac{N_A}{N}$

  ...with $N$ mutually exclusive equally likely outcomes,
  $N_A$ of which result in the occurrence of $A$. *Laplace, 1814*

- Frequentist definition: $\mathbb{P}(A) = \lim_{N\to\infty} \dfrac{N_A}{N}$

  ...relative frequency of occurrence of $A$ in infinite number of trials.

- Subjective probability: $\mathbb{P}(A)$ is a degree of belief. *de Finetti, 1930s*

  .

  ...gives meaning to $\mathbb{P}($"it will rain today"$)$, or
  $\mathbb{P}($"Patient A has disease $x$"$)$

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# The concept of probability is not as simple as you think

Nevin Climenhaga

A summary of some interpretations of probability

| | Classical | Frequentist | Subjective | Propensity |
|---|---|---|---|---|
| **Main hypothesis** | Principle of indifference | Frequency of occurrence | Degree of belief | Degree of causal connection |
| **Conceptual basis** | Hypothetical symmetry | Past data and reference class | Knowledge and intuition | Present state of system |
| **Conceptual approach** | Conjectural | Empirical | Subjective | Metaphysical |
| **Single case possible** | Yes | No | Yes | Yes |
| **Precise** | Yes | No | No | Yes |
| **Problems** | Ambiguity in principle of indifference | Circular definition | Reference class problem | Disputed concept |

"The mathematics of probability can be developed on an entirely axiomatic basis, independent of any interpretation." (wikipedia)

# Key concepts: Sample space and events

- Sample space $\mathcal{X}$ = set of possible outcomes of a random experiment.

  Examples:

  - Tossing two coins: $\mathcal{X} = \{HH, TH, HT, TT\}$
  - Roulette: $\mathcal{X} = \{1, 2, ..., 36\}$
  - Draw a card from a shuffled deck: $\mathcal{X} = \{A\clubsuit, 2\clubsuit, ..., Q\diamondsuit, K\diamondsuit\}$.

- An event $A$ is a subset of $\mathcal{X}$: $A \subseteq \mathcal{X}$ (also written $A \in 2^{\mathcal{X}}$).

  Examples:

  - "exactly one H in 2-coin toss": $A = \{TH, HT\}$.
  - "odd number in the roulette": $B = \{1, 3, ..., 35\}$.
  - "drawn a $\heartsuit$ card": $C = \{A\heartsuit, 2\heartsuit, ..., K\heartsuit\}$

# Key concepts: Sample space and events

- Sample space $\mathcal{X}$ = set of possible outcomes of a random experiment.

  (More delicate) examples:

  - Distance travelled by tossed die: $\mathcal{X} = \mathbb{R}_+$
  - Location of the next rain drop on a given square tile: $\mathcal{X} = \mathbb{R}^2$

- Properly handling the continuous case requires deeper concepts:

  - Sigma algebras, Borel sets, measurable functions, ...

...mathematically **heavier** stuff, not covered here

# Kolmogorov's Axioms for Probability

- Probability is a function that maps events $A$ into the interval $[0, 1]$.

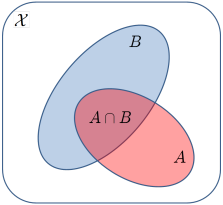
  Kolmogorov's axioms (1933) for probability

  - For any $A$, $\mathbb{P}(A) \geq 0$
  - $\mathbb{P}(\mathcal{X}) = 1$
  - If $A_1, A_2 ... \subseteq \mathcal{X}$ are disjoint events, then $\mathbb{P}\Big(\bigcup_i A_i\Big) = \sum_i \mathbb{P}(A_i)$

- From these axioms, many results can be derived.

  Examples:

  - $\mathbb{P}(\emptyset) = 0$
  - $C \subset D \;\Rightarrow\; \mathbb{P}(C) \leq \mathbb{P}(D)$
  - $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
  - $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$ (union bound)



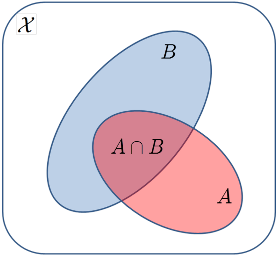
# Conditional Probability and Independence

- If $\mathbb{P}(B) > 0$, $\mathbb{P}(A|B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$ (conditional prob. of $A$, given $B$)

- ...satisfies all of Kolmogorov's axioms:

  - For any $A \subseteq \mathcal{X}$, $\mathbb{P}(A|B) \geq 0$
  - $\mathbb{P}(\mathcal{X}|B) = 1$
  - If $A_1$, $A_2$, ... $\subseteq \mathcal{X}$ are disjoint,

$$\mathbb{P}\left(\bigcup_i A_i \middle| B\right) = \sum_i \mathbb{P}(A_i|B)$$




- Independence: $A$, $B$ are independent ($A \perp\!\!\!\perp B$):

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B).$$

# Conditional Probability and Independence

- If $\mathbb{P}(B) > 0$, $\qquad \mathbb{P}(A|B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

- Events $A$, $B$ are independent $(A \perp\!\!\!\perp B) \;\Leftrightarrow\; \mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$.

- Relationship with conditional probabilities:

$$A \perp\!\!\!\perp B \;\Leftrightarrow\; \mathbb{P}(A|B) = \mathbb{P}(A)$$

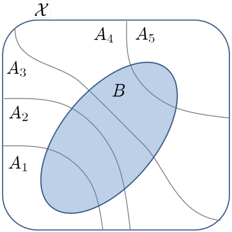
- Example: $\mathcal{X} =$ "52 cards", $A = \{4\heartsuit, 4\clubsuit, 4\diamondsuit, 4\spadesuit\}$, and $B = \{A\heartsuit, 2\heartsuit, ..., K\heartsuit\}$; then, $\mathbb{P}(A) = 1/13$, $\mathbb{P}(B) = 1/4$

$$\begin{aligned}
\mathbb{P}(A \cap B) &= \mathbb{P}(\{4\heartsuit\}) = \frac{1}{52} \\
\mathbb{P}(A)\,\mathbb{P}(B) &= \frac{1}{13}\,\frac{1}{4} = \frac{1}{52} \\
\mathbb{P}(A|B) &= \mathbb{P}(\text{“}4\text{”}\,|\,\text{“}\heartsuit\text{”}) = \frac{1}{13} = \mathbb{P}(A)
\end{aligned}$$

# Bayes Theorem

- Law of total probability: if $A_1, ..., A_n$ are a partition of $\mathcal{X}$

$$\mathbb{P}(B) = \sum_i \mathbb{P}(B|A_i)\mathbb{P}(A_i)$$
$$= \sum_i \mathbb{P}(B \cap A_i)$$




- Bayes' theorem: if $\{A_1, ..., A_n\}$ is a partition of $\mathcal{X}$

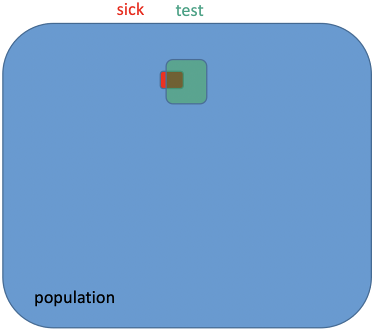
$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B \cap A_i)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A_i)\,\mathbb{P}(A_i)}{\sum_j \mathbb{P}(B|A_j)\mathbb{P}(A_j)}$$

# Bayesian inference

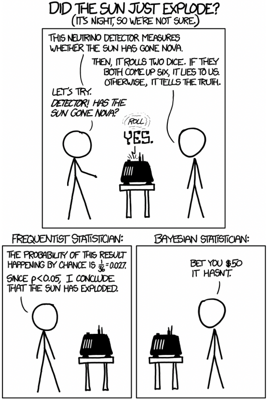
Bayes (1763)

Laplace (1812)

$$P(\text{sick} \mid \text{test}) = \frac{P(\text{test}, \text{sick})}{P(\text{test})} = \frac{\overbrace{P(\text{test} \mid \text{sick})}^{\text{sensitivity}}\ \overbrace{P(\text{sick})}^{\text{prevalence}}}{P(\text{test} \mid \text{sick})\, P(\text{sick}) + \underbrace{P(\text{test} \mid \text{not sick})}_{\text{false positive}}\ \underbrace{P(\text{not sick})}_{(1-\text{prevalence})}}$$

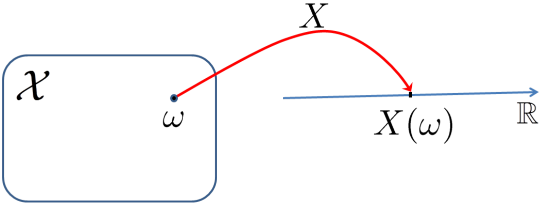



 = prevalence = P(sick)



# Random Variables

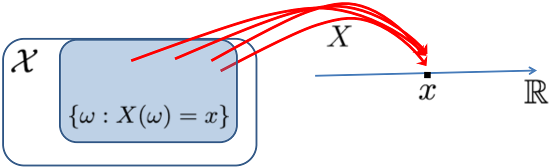
- A (real) random variable (RV) is a function: $X : \mathcal{X} \rightarrow \mathbb{R}$



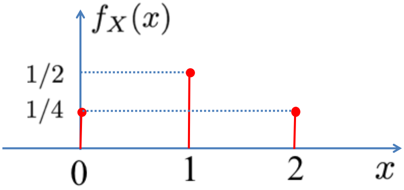

  - Discrete RV: range of $X$ is countable (*e.g.*, $\mathbb{N}$ or $\{0, 1\}$)
  - Continuous RV: range of $X$ is uncountable (*e.g.*, $\mathbb{R}$ or $[0,\, 1]$)
  - Example: number of heads in tossing two coins,
    $\mathcal{X} = \{HH, HT, TH, TT\}$,
    $X(HH) = 2$, $X(HT) = X(TH) = 1$, $X(TT) = 0$.
    Range of $X = \{0, 1, 2\}$.
  - Example: distance traveled by a tossed coin; range of $X = \mathbb{R}_+$.

# Discrete Random Variables

- Probability mass function: $f_X(x) = \mathbb{P}(\{\omega \in \mathcal{X} :\ X(\omega) = x\})$




- Example: number of heads in tossing 2 coins; range$(X) = \{0,\, 1,\, 2\}$.



# Important Discrete Random Variables

- Uniform: $X \in \{x_1, ..., x_K\}$, pmf $f_X(x_i) = 1/K$.

  Example: a fair roulette $X \in \{1, ..., 36\}$, with $f_X(x) = 1/36$

  Example: a fair die $X \in \{1, ..., 6\}$, with $f_X(x) = 1/6$

- Bernoulli RV: $X \in \{0, 1\}$, pmf $f_X(x) = \left\{ \begin{array}{ccc} p & \Leftarrow & x = 1 \\ 1 - p & \Leftarrow & x = 0 \end{array} \right.$

  Compact form: $f_X(x) = p^x (1-p)^{1-x}$.

  Example: a coin toss; heads = 0, tails = 1

  fair, if $p = 1/2$; unfair, if $p \neq 1/2$

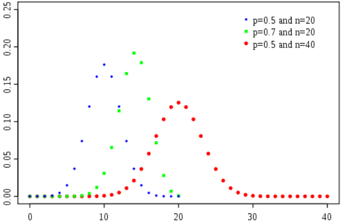
# Important Discrete Random Variables

- Binomial RV: $X \in \{0, 1, ..., n\}$ (sum of $n$ Bernoulli RVs)

$$f_X(x) = \text{Binomial}(x; n, p) = \binom{n}{x} p^x \, (1-p)^{(n-x)}$$

Binomial coefficients
("$n$ choose $x$"):

$$\binom{n}{x} = \frac{n!}{(n-x)!\,x!}$$




Example: number of heads in $n$ coin tosses.

# Other Important Discrete Random Variables

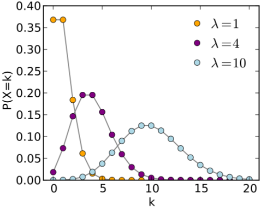
- Geometric($p$): $X \in \mathbb{N}$, pmf $f_X(x) = p(1-p)^{x-1}$.

  Example: number of coin tosses until first heads.

- Poisson($\lambda$):

$$X \in \mathbb{N} \cup \{0\},$$

$$\text{pmf } f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$



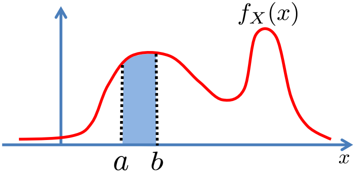

"...probability of the number of independent occurrences in a fixed (time/space) interval, if these occurrences have known average rate"

Examples: number of rain drops per second on a given area, number of calls per hour in a call center, number of tweets per day by DT, ...

# Continuous Random Variables

- Probability density function (pdf, continuous RV): $f_X(x)$

$$\int_{-\infty}^{\infty} f_X(x) = 1 \qquad \mathbb{P}(X \in [a,b]) = \int_a^b f_X(x)\,dx$$

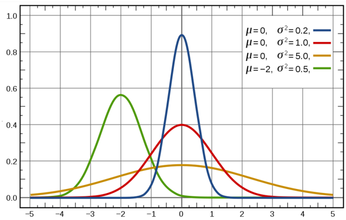


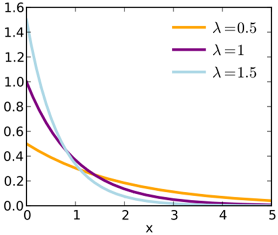

- Notice: $\mathbb{P}(X = c) = 0$

# Important Continuous Random Variables

- Uniform: $f_X(x) = \text{Uniform}(x; a, b) = \begin{cases} \frac{1}{b-a} & \Leftarrow \quad x \in [a,\, b] \\ 0 & \Leftarrow \quad x \notin [a,\, b] \end{cases}$

- Gaussian: $f_X(x) = \mathcal{N}(x; \mu, \sigma^2) = \dfrac{1}{\sqrt{2\,\pi\,\sigma^2}} e^{-\frac{(x-\mu)^2}{2\,\sigma^2}}$




- Exponential: $f_X(x) = \text{Exp}(x; \lambda) = \begin{cases} \lambda e^{-\lambda\, x} & \Leftarrow \quad x \geq 0 \\ 0 & \Leftarrow \quad x < 0 \end{cases}$

# Expectation of (Real) Random Variables

- Expectation: $\mathbb{E}(X) = \begin{cases} \displaystyle\sum_i x_i \, f_X(x_i) & X \in \{x_1, ...x_K\} \subset \mathbb{R} \\ \displaystyle\int_{-\infty}^{\infty} x \, f_X(x) \, dx & X \text{ continuous} \end{cases}$

- Example: Bernoulli, $f_X(x) = p^x \, (1-p)^{1-x}$, for $x \in \{0, \, 1\}$.

  $$\mathbb{E}(X) = 0 \, (1-p) + 1 \, p = p.$$

- Example: Binomial, $f_X(x) = \binom{n}{x} p^x \, (1-p)^{n-x}$, for $x \in \{0, ..., n\}$.

  $$\mathbb{E}(X) = n \, p.$$

- Example: Gaussian, $f_X(x) = \mathcal{N}(x; \mu, \sigma^2)$. $\quad \mathbb{E}(X) = \mu$.

- Linearity of expectation:

  $$\mathbb{E}(\alpha \, X + \beta \, Y) = \alpha \, \mathbb{E}(X) + \beta \, \mathbb{E}(Y), \qquad \alpha, \beta \in \mathbb{R}$$

# Expectation of Functions of RVs

- $$\mathbb{E}(g(X)) = \begin{cases} \displaystyle\sum_i g(x_i) f_X(x_i) & X \text{ discrete, } g(x_i) \in \mathbb{R} \\ \displaystyle\int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx & X \text{ continuous} \end{cases}$$

- Example: variance, $\text{var}(X) = \mathbb{E}\Big(\big(X - \mathbb{E}(X)\big)^2\Big) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

- Example: Bernoulli variance, $\mathbb{E}(X^2) = \mathbb{E}(X) = p$ , thus $\text{var}(X) = p(1-p)$.

- Example: Gaussian variance, $\mathbb{E}\big((X-\mu)^2\big) = \sigma^2$.

- Probability as expectation of indicator, $\mathbf{1}_A(x) = \begin{cases} 1 & \Leftarrow & x \in A \\ 0 & \Leftarrow & x \notin A \end{cases}$

$$\mathbb{P}(X \in A) = \int_A f_X(x)\, dx = \int \mathbf{1}_A(x)\, f_X(x)\, dx = \mathbb{E}(\mathbf{1}_A(X))$$

# The importance of the Gaussian

# The importance of the Gaussian

Take $n$ independent RVs $X_1, ..., X_n$, with $\mathbb{E}[X_i] = \mu_i$ and $\text{var}(X_i) = \sigma_i^2$

- Their sum, $Y_n = \sum_{i=1}^{n} X_i$ satisfies:

$$\mathbb{E}[Y_n] = \sum_{i=1}^{n} \mu_i \equiv \mu \qquad\qquad \text{var}(Y_n) = \sum_i \sigma_i^2 \equiv \sigma^2$$

- Let $Z_n = \dfrac{Y_n - \mu}{\sigma}$, thus $\mathbb{E}[Z_n] = 0$ and $\text{var}(Z_n) = 1$

- Central limit theorem: under mild conditions,

$$\lim_{n\to\infty} Z_n \sim \mathcal{N}(0,1)$$

# Two (or More) Random Variables

- Joint pmf of two discrete RVs: $f_{X,Y}(x,y) = \mathbb{P}(X = x \wedge Y = y)$.

  Extends trivially to more than two RVs.

- Joint pdf of two continuous RVs: $f_{X,Y}(x,y)$, such that

$$\mathbb{P}\big((X,Y) \in A\big) = \iint_A f_{X,Y}(x,y)\,dx\,dy, \qquad A \in \sigma(\mathbb{R}^2)$$

  Extends trivially to more than two RVs.

- Marginalization: $f_Y(y) = \begin{cases} \displaystyle\sum_x f_{X,Y}(x,y), & \text{if } X \text{ is discrete} \\ \displaystyle\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx, & \text{if } X \text{ continuous} \end{cases}$

- Independence:

$$X \perp\!\!\!\perp Y \;\Leftrightarrow\; f_{X,Y}(x,y) = f_X(x)\,f_Y(y) \;\underset{\not\Leftarrow}{\Rightarrow}\; \mathbb{E}(X\,Y) = \mathbb{E}(X)\,\mathbb{E}(Y).$$

# Conditionals and Bayes' Theorem

- Conditional pmf (discrete RVs):

$$f_{X|Y}(x|y) = \mathbb{P}(X = x|Y = y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

- Conditional pdf (continuous RVs): $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$

  ...the meaning is technically delicate.

- Bayes' theorem: $f_{X|Y}(x|y) = \dfrac{f_{Y|X}(y|x)\, f_X(x)}{f_Y(y)}$ (pdf or pmf).

- Also valid in the mixed case (*e.g.*, $X$ continuous, $Y$ discrete).

# Joint, Marginal, and Conditional Probabilities: An Example

- A pair of binary variables $X, Y \in \{0, 1\}$, with joint pmf:

| $f_{X,Y}(x,y)$ | $Y = 0$ | $Y = 1$ |
|---|---|---|
| $X = 0$ | 1/5 | 2/5 |
| $X = 1$ | 1/10 | 3/10 |

- Marginals: $f_X(0) = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}$, $\quad f_X(1) = \frac{1}{10} + \frac{3}{10} = \frac{4}{10}$,

$$f_Y(0) = \frac{1}{5} + \frac{1}{10} = \frac{3}{10}, \quad f_Y(1) = \frac{2}{5} + \frac{3}{10} = \frac{7}{10}.$$

- Conditional probabilities:

| $f_{X\|Y}(x\|y)$ | $Y = 0$ | $Y = 1$ |
|---|---|---|
| $X = 0$ | 2/3 | 4/7 |
| $X = 1$ | 1/3 | 3/7 |

| $f_{Y\|X}(y\|x)$ | $Y = 0$ | $Y = 1$ |
|---|---|---|
| $X = 0$ | 1/3 | 2/3 |
| $X = 1$ | 1/4 | 3/4 |

# An Important Multivariate RV: Multinomial

- Multinomial: $X = (X_1, ..., X_K)$, $X_i \in \{0, ..., n\}$, s.t. $\sum_i X_i = n$,

$$f_X(x_1, ..., x_K) = \begin{cases} \binom{n}{x_1\ x_2\ \cdots\ x_K} p_1^{x_1}\, p_2^{x_2} \cdots p_K^{x_K} & \Leftarrow \quad \sum_i x_i = n \\ 0 & \Leftarrow \quad \sum_i x_i \neq n \end{cases}$$

$$\binom{n}{x_1\ x_2\ \cdots\ x_K} = \frac{n!}{x_1!\, x_2! \cdots x_K!}$$

Parameters: $p_1, ..., p_K \geq 0$, such that $\sum_i p_i = 1$.

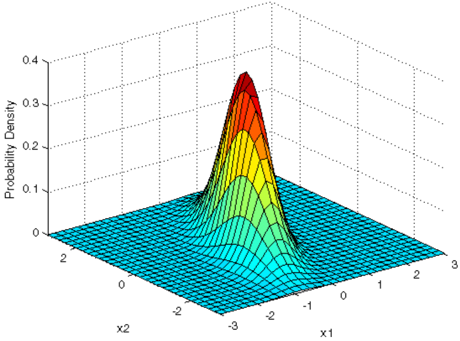
- Generalizes the binomial from binary to $K$-classes.
- Example: tossing $n$ independent fair dice, $p_1 = \cdots = p_6 = 1/6$.
  $x_i =$ number of outcomes with $i$ dots (of course, $\sum_i x_i = n$)
- Example: bag of words (BoW) multinomial model with vocabulary of $K$ words

# An Important Multivariate RV: Gaussian

- Multivariate Gaussian: $X \in \mathbb{R}^n$,

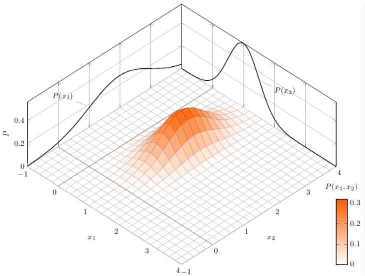
$$f_X(x) = \mathcal{N}(x;\mu,C) = \frac{1}{\sqrt{\det(2\,\pi\,C)}}\exp\left(-\frac{1}{2}(x-\mu)^T C^{-1}(x-\mu)\right)$$

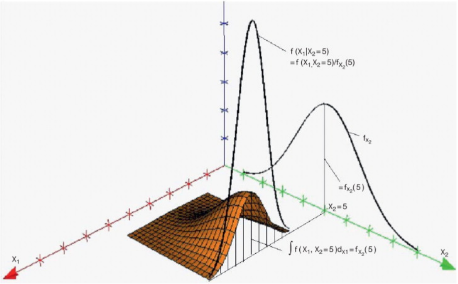
- Parameters: vector $\mu \in \mathbb{R}^n$ and matrix $C \in \mathbb{R}^{n\times n}$.
  Expected value: $\mathbb{E}(X) = \mu$. Meaning of $C$: later.



# Key Properties of Multivariate Gaussian

- Marginals are Gaussian.
- Conditionals are Gaussian.



# Transformations

$X \sim f_X$ and $Y = g(X) \;\Rightarrow\; f_Y =?$

- Discrete case:
  $f_Y(y) = \mathbb{P}(g(X) = y) = \mathbb{P}\big(\{x : g(x) = y\}\big) = \mathbb{P}\big(g^{-1}(y)\big)$

- Continuous case (for $g$ strictly monotonic, thus invertible):

$$f_Y(y) = f_X\left(g^{-1}(y)\right)\Big|\frac{d\, g^{-1}(y)}{d\, y}\Big|$$

- Continuous multivariate case (invertible):

$$f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\text{det} J_{g^{-1}}(y)\right|$$

where $\text{det} J_{g^{-1}}(y)$ is the determinant of the Jacobian of $g^{-1}$ at $y$.

# Covariance, Correlation, and all that...

- Covariance between two RVs:

$$\text{cov}(X,Y) = \mathbb{E}\Big[\big(X - \mathbb{E}(X)\big)\,\big(Y - \mathbb{E}(Y)\big)\Big] \;=\; \mathbb{E}(X\,Y) - \mathbb{E}(X)\,\mathbb{E}(Y)$$

- Relationship with variance: $\text{var}(X) = \text{cov}(X,X)$.

- Correlation: $\text{corr}(X,Y) = \rho(X,Y) = \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)}\sqrt{\text{var}(Y)}} \in [-1,\,1]$

- $X \perp\!\!\!\perp Y \;\Leftrightarrow\; f_{X,Y}(x,y) = f_X(x)\,f_Y(y) \;\underset{\not\Leftarrow}{\overset{\Rightarrow}{}}\; \text{cov}(X,\,Y) = 0.$

- Covariance matrix of multivariate RV, $X \in \mathbb{R}^n$:

$$\text{cov}(X) = \mathbb{E}\Big[\big(X - \mathbb{E}(X)\big)\big(X - \mathbb{E}(X)\big)^T\Big] = \mathbb{E}(X\,X^T) - \mathbb{E}(X)\mathbb{E}(X)^T$$

- Covariance of Gaussian RV, $f_X(x) = \mathcal{N}(x;\mu,C) \;\Rightarrow\; \text{cov}(X) = C$

# More on Expectations and Covariances

Let $A \in \mathbb{R}^{n\times n}$ be a matrix and $a \in \mathbb{R}^n$ a vector.

- If $\mathbb{E}(X) = \mu$ and $Y = AX$, then $\mathbb{E}(Y) = A\mu$;
- If $\mathbb{E}(X) = \mu$ and $Y = X + \gamma$, then $\mathbb{E}(Y) = \mu + \gamma$;
- If $\text{cov}(X) = C$ and $Y = AX$, then $\text{cov}(Y) = ACA^T$;
- If $\text{cov}(X) = C$ and $Y = a^T X \in \mathbb{R}$, then $\text{var}(Y) = a^T C a \geq 0$;
- If $\text{cov}(X) = C$ and $Y = C^{-1/2}X$, then $\text{cov}(Y) = I$;

Combining the 2-nd and the 5-th facts: standardization:

$$\mathbb{E}(X) = \mu,\;\; \text{cov}(X) = C,\;\; Y = C^{-\frac{1}{2}}(X-\mu) \quad\Rightarrow\quad \mathbb{E}(Y) = 0,\;\; \text{cov}(Y) = I$$

Combining the 2-nd and the 3-rd facts: reparametrization trick:

$$\mathbb{E}(X) = 0,\;\; \text{cov}(X) = I,\;\; Y = AX+\mu \quad\Rightarrow\quad \mathbb{E}(Y) = \mu,\;\; \text{cov}(Y) = AA^T$$

# Exponential Families

A pdf or pmf $f_X(x|\eta)$, with parameter(s) $\eta$, for $X \in \mathcal{X}$, is in an exponential family if

$$f_X(x|\eta) = \frac{1}{Z(\eta)}\, h(x)\, \exp\big(\eta^T \phi(x)\big)$$

where $\eta^T \phi(x) = \sum_j \eta_j \phi_j(x)$ and

$$Z(\eta) = \int_{\mathcal{X}} h(x)\, \exp\big(\eta^T \phi(x)\big)\, dx.$$

- Canonical parameter(s): $\eta$
- Sufficient statistics: $\phi(x)$
- Partition function: $Z(\eta)$

Examples: Bernoulli, Poisson, binomial, multinomial, Gaussian, exponential, beta, Dirichlet, Laplacian, log-normal, Wishart, ...

# Exponential Families

$$f_X(x|\eta) = \frac{1}{Z(\eta)}\, h(x)\, \exp\big(\eta^T \phi(x)\big)$$

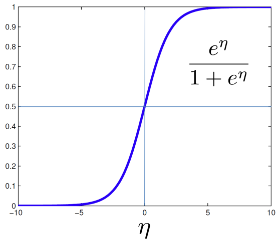
- Example: Bernoulli pmf $f_X(x) = p^x (1-p)^{1-x}$,

$$f_X(x) = \exp\big(x \log p + (1-x)\log(1-p)\big) = (1-p)\exp\big(x \log \tfrac{p}{1-p}\big),$$

thus $\eta = \log \frac{p}{1-p}$, $\phi(x) = x$, $Z(\eta) = 1 + e^\eta$, and $h(x) = 1$.

Notice that $p = \frac{e^\eta}{1+e^\eta}$

(logistic transformation)



# More on Exponential Families

- Independent identically distributed (i.i.d.) observations:

$$X_1, ..., X_m \sim f_X(x|\eta) = \frac{1}{Z(\eta)} h(x) \exp\big(\eta^T \phi(x)\big)$$

then

$$f_{X_1,...,X_m}(x_1, ..., x_m|\eta) = \frac{1}{Z(\eta)^m} \left(\prod_{j=1}^m h(x_i)\right) \exp\left(\eta^T \sum_{j=1}^m \phi(x_j)\right)$$

- Expected sufficient statistics:

$$\frac{d \log Z(\eta)}{d\eta} = \frac{\frac{dZ(\eta)}{d\eta}}{Z(\eta)} = \frac{1}{Z(\eta)} \int \phi(x) h(x) \exp\big(\eta^T \phi(x)\big)\, dx = \mathbb{E}\big(\phi(X)\big)$$

可以用Bernoulli分布验证

# Important Inequalities

- Markov's ineqality: if $X \geq 0$ is an RV with expectation $\mathbb{E}(X)$, then

$$\mathbb{P}(X > t) \leq \frac{\mathbb{E}(X)}{t}$$

Simple proof:

$$t\,\mathbb{P}(X > t) = \int_t^\infty t\, f_X(x)\, dx \leq \int_t^\infty x\, f_X(x)\, dx = \mathbb{E}(X) - \underbrace{\int_0^t x\, f_X(x)\, dx}_{\geq 0} \leq \mathbb{E}(X)$$

- Chebyshev's inequality: $\mu = \mathbb{E}(Y)$ and $\sigma^2 = \mathrm{var}(Y)$, then

$$\mathbb{P}(|Y - \mu| \geq s) \leq \frac{\sigma^2}{s^2}$$

...simple corollary of Markov's inequality, with $X = |Y - \mu|^2$, $t = s^2$

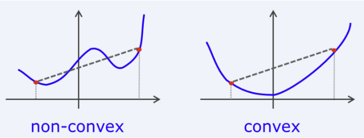
# Important Inequalities

- Cauchy-Schwartz's inequality for RVs:

$$\mathbb{E}(|X\,Y|) \leq \sqrt{\mathbb{E}(X^2)\mathbb{E}(Y^2)}$$

- Recall that a real function $g$ is convex if, for any $x, y$, and $\alpha \in [0,1]$

$$g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y)$$




Jensen's inequality: if $g$ is a real convex function, then

$$g(\mathbb{E}(X)) \leq \mathbb{E}(g(X))$$

Examples: $\mathbb{E}(X)^2 \leq \mathbb{E}(X^2) \;\Rightarrow\; \text{var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \geq 0.$
$\mathbb{E}(\log X) \leq \log \mathbb{E}(X)$, for $X$ a positive RV.

# Information, entropy, and all that...

Entropy of a discrete RV $X \in \{1, ..., K\}$: $\boxed{H(X) = -\sum_{x=1}^{K} f_X(x) \log f_X(x)}$

- Positivity: $H(X) \geq 0$ ;
  $H(X) = 0 \;\Leftrightarrow\; f_X(i) = 1$, for exactly one $i \in \{1, ..., K\}$.
- Upper bound: $H(X) \leq \log K$ ;
  $H(X) = \log K \;\Leftrightarrow\; f_X(x) = 1/K$, for all $x \in \{1, ..., K\}$
- Measure of uncertainty/randomness of $X$
- With $\log_2$, units are bits/symbol
- Central role in information/coding theory: lower bound on expected number of bits to code $X$
- Widely used: physics, biological sciences (computational biology, neurosciences, ecology, ...), economics, finances, social sciences, ...

# Entropy and all that...

Continuous RV $X$, differential entropy: $\boxed{h(X) = -\int f_X(x) \log f_X(x)\, dx}$

- $h(X)$ can be positive or negative (unlike in the discrete case)

  Example: for $f_X(x) = \text{Uniform}(x; a, b)$,

$$h(X) = \log(b - a).$$

- Gaussian upper bound: $f_X(x) = \mathcal{N}(x; \mu, \sigma^2)$, then

$$h(X) = \frac{1}{2} \log(2\pi e \sigma^2).$$

For any RV $Y$ with $\text{var}(Y) = \sigma^2$, then $h(Y) \leq \frac{1}{2} \log(2\pi e \sigma^2)$.

...yet another reason for why the Gaussian is important.

# Kullback-Leibler divergence

Kullback-Leibler divergence (KLD) between two pmf:

$$D(f_X \| g_X) = \sum_{x=1}^{K} f_X(x) \log \frac{f_X(x)}{g_X(x)}$$

Positivity: $D(f_X \| g_X) \geq 0$
$D(f_X \| g_X) = 0 \;\Leftrightarrow\; f_X(x) = g_X(x)$, for $x \in \{1, ..., K\}$

KLD between two pdf:

$$D(f_X \| g_X) = \int f_X(x) \log \frac{f_X(x)}{g_X(x)}\, dx$$

Positivity: $D(f_X \| g_X) \geq 0$
$D(f_X \| g_X) = 0 \;\Leftrightarrow\; f_X(x) = g_X(x)$, almost everywhere

Issues: not symmetric; $D(f_X \| g_X) = +\infty$ if $g_X(x) = 0$ and $f_X(x) \neq 0$

# Mutual information

Mutual information (MI) between two random variables:

$$I(X;Y) = D\big(f_{X,Y} \| f_X\, f_Y\big)$$

Positivity: $I(X;Y) \geq 0$
$I(X;Y) = 0 \;\Leftrightarrow\; X, Y$ are independent.

MI $=$ measure of dependency between two random variables

MI $=$ number of bits of information that $X$ has about $Y$

Bound: $I(X;Y) \leq \min\{H(X), H(Y)\}$

Deterministic function: if $Y = \phi(X)$, then $I(X;Y) = H(Y) \leq H(X)$

# Recommended Reading (Probability and Statistics)


- A. Maleki and T. Do, "Review of Probability Theory", Stanford University, 2017 (`https://tinyurl.com/pz7p9g5`)
- K. Murphy, "Machine Learning: A Probabilistic Perspective", MIT Press, 2012 (Chapter 2).
- L. Wasserman, "All of Statistics: A Concise Course in Statistical Inference", Springer, 2004.

Part II: Algebra and a Few Other Things

# Notation: Matrices and Vectors

- $A \in \mathbb{R}^{m\times n}$ is a matrix with $m$ rows and $n$ columns.

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix}.$$

- $x \in \mathbb{R}^n$ is a vector with $n$ components,

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

- A (column) vector is a matrix with $n$ rows and $1$ column.

- A matrix with $1$ row and $n$ columns is called a row vector.

# Matrix Transpose and Products

- Given matrix $A \in \mathbb{R}^{m\times n}$, its transpose $A^T$ is such that $(A^T)_{i,j} = A_{j,i}$.
- A matrix $A$ is symmetric if $A^T = A$.
- Given matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$, their product is

$$C = A\,B \ \in \mathbb{R}^{m\times p} \quad \text{where} \quad C_{i,j} = \sum_{k=1}^{n} A_{i,k}\, B_{k,j}$$

- Inner product between vectors $x, y \in \mathbb{R}^n$:

$$\langle x, y\rangle = x^T y = y^T x = \sum_{i=1}^{n} x_i\, y_i \in \mathbb{R}.$$

- Outer product: $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$: $x\, y^T \in \mathbb{R}^{n\times m}$, where

$$(x\, y^T)_{i,\,j} = x_i\, y_j$$

# Properties of Matrix Products and Transposes

- Given matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$, their product is

$$C = A\,B \ \in \mathbb{R}^{m\times p} \quad \text{where} \;\; C_{i,j} = \sum_{k=1}^{n} A_{i,k}\, B_{k,j}$$

- Matrix product is associative: $(AB)C = A(BC)$.
- In general, matrix product is not commutative: $AB \neq BA$.
- Transpose of product: $(A\,B)^T = B^T A^T$.
- Transpose of sum: $(A+B)^T = A^T + B^T$.

# Special Matrices

- The identity matrix $I \in \mathbb{R}^{n\times n}$ is a square matrix such that

$$I_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \qquad I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

- Neutral element of matrix product: $A\,I = I\,A = A$.
- Diagonal matrix: $(i \neq j) \Rightarrow A_{i,j} = 0$.
- Upper triangular matrix: $(j < i) \Rightarrow A_{i,j} = 0$.
- Lower triangular matrix: $(j > i) \Rightarrow A_{i,j} = 0$.

# Eigenvalues, eigenvectors, determinant, trace

- A vector $x \in \mathbb{R}^n$ is an eigenvector of matrix $A \in \mathbb{R}^{n\times n}$ if

$$A\,x = \lambda\,x,$$

where $\lambda \in \mathbb{R}$ is the corresponding eigenvalue.

- The eigenvalues of a diagonal matrix are the elements in the diagonal. (quiz: what are the eigenvectors?)

- Matrix trace: $\text{trace}(A) = \sum_i A_{i,i} = \sum_i \lambda_i$

- Matrix determinant: $|A| = \det(A) = \prod_i \lambda_i$

- Properties of determinant: $|AB| = |A||B|$, $|A^T| = |A|$, $|\alpha\, A| = \alpha^n |A|$

- Properties of the trace: $\text{trace}(A+B) = \text{trace}(A) + \text{trace}(B)$, $\text{trace}(ABC) = \text{trace}(CAB) = \text{trace}(BCA)$ (cyclic permutations)

# Matrix Inverse

- Matrix $A \in \mathbb{R}^{n\times n}$ in invertible if there is $B \in \mathbb{R}^{n\times n}$ s.t. $AB = BA = I$.
- ...matrix $B$, such that $AB = BA = I$, denoted $B = A^{-1}$.
- Matrix $A \in \mathbb{R}^{n\times n}$ is invertible $\Leftrightarrow \det(A) \neq 0$.
- Determinant of inverse: $\det(A^{-1}) = \dfrac{1}{\det(A)}$.
- Solving system $Ax = b$, if $A$ is invertible: $x = A^{-1}b$.
- Properties: $(A^{-1})^{-1} = A, \;\; (A^{-1})^T = (A^T)^{-1}, \;\; (A\,B)^{-1} = B^{-1}A^{-1}$
- There are many algorithms to compute $A^{-1}$; general case, computational cost $O(n^3)$.

# Quadratic Forms and Positive (Semi-)Definite Matrices

- Given matrix $A \in \mathbb{R}^{n \times n}$ and vector $x \in \mathbb{R}^n$,

$$x^T A\, x = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j}\, x_i\, x_j \ \in \ \mathbb{R}$$

  is called a quadratic form.

- A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive semi-definite (PSD) if, for any $x \in \mathbb{R}^n$, $x^T A\, x \geq 0$.

- A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive definite (PD) if, for any $x \in \mathbb{R}^n$, $(x \neq 0) \Rightarrow x^T A\, x > 0$.

- Matrix $A \in \mathbb{R}^{n \times n}$ is PSD $\Leftrightarrow$ all $\lambda_i(A) \geq 0$.

- Matrix $A \in \mathbb{R}^{n \times n}$ is PD $\Leftrightarrow$ all $\lambda_i(A) > 0$.

# A Bit More Formal: Vector Spaces

- A vector space over a field $\mathbb{F}$ (*e.g.*, $\mathbb{R}$) is a set $\mathbb{V}$ and a pair of operations, $+ : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ and $\cdot : \mathbb{F} \times \mathbb{V} \to \mathbb{V}$, that satisfy the following axioms, $\forall x, y, z \in \mathbb{V}$ and $\forall \alpha, \beta \in \mathbb{F}$:
  - ✓ $+$ is associative and commutative;
  - ✓ $\exists 0 \in \mathbb{V}$, such that $0 + x = x$;
  - ✓ $\exists -x \in \mathbb{V}$, such that $-x + x = 0$;
  - ✓ $\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x$;
  - ✓ $1 \cdot x = x$, where $1 \in \mathbb{F}$ is such that $1 \cdot \alpha = \alpha$;
  - ✓ $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$;
  - ✓ $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$.
- Elements of $\mathbb{V}$ are called vectors; elements of $\mathbb{F}$ are scalars.
- Standard compact notation: $\alpha\, x \equiv \alpha \cdot x$.

# Vector Space: Examples

- "Usual vectors" $(\mathbb{R}^n, +, \cdot)$ over field $\mathbb{R}$
  - ✓ $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$, $x + y = (x_1 + y_1, ..., x_n + y_n)$;
  - ✓ $x = (x_1, ..., x_n)$, $\alpha\, x = (\alpha\, x_1, ..., \alpha\, x_n)$
- Real matrices $(\mathbb{R}^{m\times n}, +, \cdot)$ over field $\mathbb{R}$
  - ✓ usual matrix addition and multiplication by scalar;
- Complex matrices $(\mathbb{C}^{m\times n}, +, \cdot)$ over field $\mathbb{C}$ (complex numbers).
- Binary vectors $(\{0,1\}^n, +, \cdot)$ over $GF(2) = \{0,1\}$ (Galois field),
  - ✓ $+$ is *modulo-2* addition: $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1$, $1 + 1 = 0$.
  - ✓ $\cdot$ is standard multiplication: $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$, $1 \cdot 1 = 1$.
- Set of all functions $f : \Omega \rightarrow \mathbb{R}$, with point-wise addition and multiplication, is a vector space over $\mathbb{R}$.

# Norm on Vector Space $\mathbb{V}$

- A norm is a function $\|\cdot\| : \mathbb{V} \to \mathbb{R}_+$ satisfying,

  $\forall x, y \in \mathbb{V}$ and $\forall \alpha \in \mathbb{R}$,

  ✓ homogeneity, $\|\alpha\, x\| = |\alpha|\; \|x\|$;

  ✓ triangle inequality, $\|x + y\| \leq \|x\| + \|y\|$;

  ✓ definiteness, $\|x\| = 0 \Leftrightarrow x = 0$.

- A seminorm may not satisfy definiteness.

- Classical example in $\mathbb{R}^n$: Euclidean norm: $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$.

- Two norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent if $\exists \alpha, \beta > 0$ such that

$$\forall x \in \mathbb{V}, \quad \alpha\|x\| \leq \|x\|' \leq \beta\|x\|;$$

  if $\mathbb{V}$ is finite-dimensional, all norms in $\mathbb{V}$ are equivalent.

# Other Norms

- The $\ell_p$ norm of a vector $x \in \mathbb{R}^n$, where $p \geq 1$,

$$\|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

  Notable cases:

  - $\ell_2$ (Euclidean) norm.
  - $\ell_1$ norm, $\|x\|_1 = \sum_i |x_i|$.
  - $\ell_\infty$ norm, $\lim_{p\to\infty} \|x\|_p = \max\{|x_1|, ..., |x_n|\} \equiv \|x\|_\infty$
  - $\ell_0$ "norm" (not a norm), $\lim_{p\to 0} \|x\|_p = \#\{i : x_i \neq 0\} \equiv \|x\|_0$

- Some equivalences:

$$\begin{array}{ccccc} \|x\|_2 & \leq & \|x\|_1 & \leq & \sqrt{n}\,\|x\|_2 \\ \|x\|_\infty & \leq & \|x\|_2 & \leq & \sqrt{n}\,\|x\|_\infty \\ \|x\|_\infty & \leq & \|x\|_1 & \leq & n\,\|x\|_\infty \end{array}$$

# Inner Product on Vector Space $\mathbb{V}$ Over $\mathbb{R}$

- An inner product is a function $\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$, satisfying, $\forall x, y \in \mathbb{V}$ and $\forall \alpha \in \mathbb{R}$,
  - ✓ symmetry, $\langle x, y \rangle = \langle y, x \rangle$
  - ✓ (bi)linearity, $\langle \alpha x + \beta z, y \rangle = \alpha \langle x, y \rangle + \beta \langle z, y \rangle$;
  - ✓ definiteness, $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$.
- Standard inner product in $\mathbb{R}^n$: $\langle x, y \rangle = \sum_{i=1}^{n} x_i \, y_i = x^T y$
- Also an inner product in $\mathbb{R}^n$: $\langle x, y \rangle = x^T M y$, where $M$ is PD
- Norm (is it?) induced by an inner product $\|x\| = \sqrt{\langle x, x \rangle}$
  - ✓ for $\langle x, y \rangle = x^T y$, then $\|x\|^2 = x^T x = \sum_{i=1}^{n} x_i^2$ (Euclidean norm)
  - ✓ for $\langle x, y \rangle = x^T M y$, then $\|x\|_M^2 = x^T M x$ (Mahalanobis norm)

# Key Properties of Inner Products

- If $\|\cdot\|$ is induced by inner product $\langle\cdot,\cdot\rangle$ (that is $\|x\|^2 = \langle x, x\rangle$), then

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$$

- Cauchy—Schwarz inequality: $\boxed{|\langle x, y\rangle| \leq \|x\|\,\|y\|}$

  "Des démonstrations qui font boum!" ("Proofs that make boom!")
  [Jean-Baptiste Hiriart-Urruty]

$$0 \leq \frac{1}{2}\left\|\frac{x}{\|x\|} \pm \frac{y}{\|y\|}\right\|^2 = 1 \pm \frac{\langle x, y\rangle}{\|x\|\,\|y\|} \Leftrightarrow \left\{\begin{array}{r} \langle x, y\rangle \leq \|x\|\,\|y\| \\ -\langle x, y\rangle \leq \|x\|\,\|y\| \end{array}\right.$$

- Corollary: $\|\cdot\|$ is indeed a norm, as it satisfies the triangle inequality:

$$\|x+y\|^2 \leq \|x\|^2 + \|y\|^2 + 2|\langle x, y\rangle| \leq \|x\|^2 + \|y\|^2 + 2\|x\|\,\|y\| = \big(\|x\| + \|y\|\big)^2$$

- Hilbert space: complete vector space equipped with an inner product.

# Basis and Dimension of a Vector Space $\mathbb{V}$

- Basis: collection of vectors $B = \{b_1, b_2, ...\} \subset \mathbb{V}$ satisfying:
  - ✓ linear independence: for any finite linear combination

$$\alpha_1 b_1 + ... + \alpha_m b_m = 0 \;\Rightarrow\; \alpha_1 = \cdots = \alpha_m = 0$$

  - ✓ spanning ability: any vector $v \in \mathbb{V}$ can be written as

$$v = \alpha_1 b_1 + ... + \alpha_m b_n;$$

  in other words, $\mathbb{V} = \text{span}(B)$.

- Dimension of $\mathbb{V}$: $\dim(\mathbb{V}) = \#B$
- Orthogonal basis: $i \neq j \Rightarrow \langle b_i, b_j \rangle = 0$
- Orthonormal basis: orthogonal and $\|b_i\| = 1$, $\forall\, b_i \in B$.

# Rank, Range, and Null Space

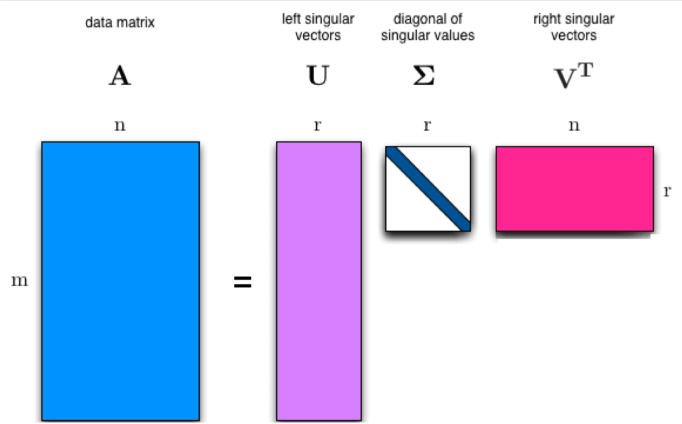
- Consider some real matrix $A \in \mathbb{R}^{m\times n}$
- Range of $A$: $\mathcal{R}(A) = \big\{y \in \mathbb{R}^m : \exists\, x \in \mathbb{R}^n \text{ such that } y = Ax\big\} \subseteq \mathbb{R}^m$
- Null space of $A$: $\mathcal{N}(A) = \big\{x \in \mathbb{R}^n : Ax = 0\big\} \subseteq \mathbb{R}^n$.
- Both $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are vector spaces.
- Dimension theorem: $\dim\big(\mathcal{R}(A)\big) + \dim\big(\mathcal{N}(A)\big) = n$
- Rank: $\text{rank}(A) = \dim\big(\mathcal{R}(A)\big) \leq \min\{m, n\}$
- $\text{rank}(A) = n - \dim\big(\mathcal{N}(A)\big)$

# Singular Value Decomposition (SVD)

- Any rank-$r$ matrix $A \in \mathbb{R}^{m \times n}$ can be written as $A = U \Lambda V^T$
  - ✓ columns of $U \in \mathbb{R}^{m \times r}$ are an orthonormal basis of $\mathcal{R}(A)$;
  - ✓ columns of $V \in \mathbb{R}^{n \times r}$ are an orthonormal basis of $\mathcal{R}(A^T)$;
  - ✓ $\Lambda = \text{diag}(\sigma_1, ..., \sigma_r)$ is a $r \times r$ diagonal matrix;
  - ✓ $\sigma_1, ..., \sigma_r$ are called singular values.
  - ✓ $\sigma_1, ..., \sigma_r$ are square roots of the eigenvalues of $A^T A$ or $A A^T$.
- Orthonormality of $U$ and $V$: $U^T U = I$ and $V^T V = I$ .
- Transposition: $A^T = (U \Lambda V^T)^T = V \Lambda U^T$.

# Singular Value Decomposition (SVD)

- $A = U \Lambda V^T$, where $U \in \mathbb{R}^{m \times r}$ and $V \in \mathbb{R}^{n \times r}$.




Picture credits: Mukesh Mithrakumar

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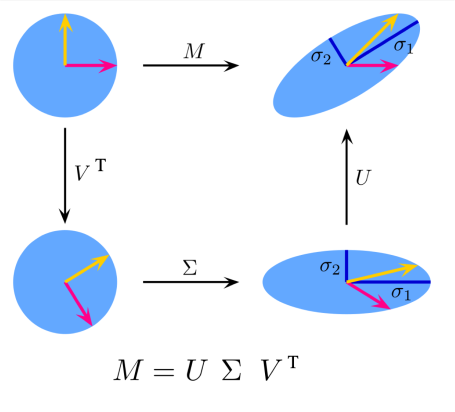
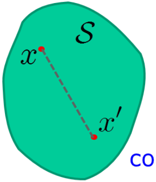



Picture credits: Wikipedia

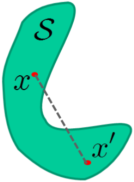
# Convex Sets

## Convex and strictly convex sets

$\mathcal{S}$ is convex if $x, x' \in \mathcal{S} \;\Rightarrow \forall \lambda \in [0,1], \;\; \lambda x + (1-\lambda)x' \in \mathcal{S}$

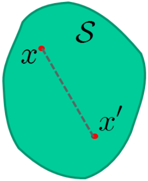



convex

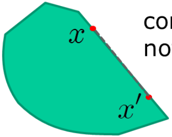



non-convex

$\mathcal{S}$ is strictly convex if $x, x' \in \mathcal{S} \;\Rightarrow \forall \lambda \in (0,1), \;\; \lambda x + (1-\lambda)x' \in \mathrm{int}(\mathcal{S})$




strictly convex




convex, but not strictly

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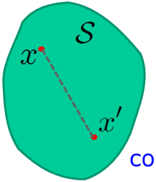
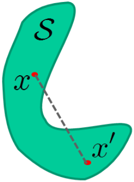



convex

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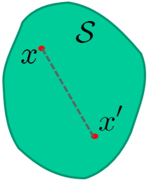


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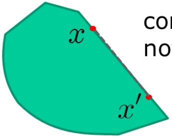



convex, but not strictly

# Convex Functions

## Convex and strictly convex functions

Extended real valued function: $f : \mathbb{R}^N \rightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$

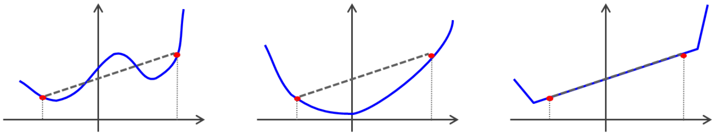
Domain of a function: $\mathrm{dom}(f) = \{x : f(x) \neq +\infty\}$

$f$ is a convex function if

$$\forall \lambda \in [0,1], x, x' \in \mathrm{dom}(f) \;\; f(\lambda x + (1-\lambda)x') \leq \lambda f(x) + (1-\lambda) f(x')$$

$f$ is a strictly convex function if

$$\forall \lambda \in (0,1), x, x' \in \mathrm{dom}(f) \;\; f(\lambda x + (1-\lambda)x') < \lambda f(x) + (1-\lambda) f(x')$$



non-convex

convex
strictly convex

convex, not strictly

# Recommended Reading

- Z. Kolter and C. Do, "Linear Algebra Review and Reference", Stanford University, 2015 (`https://tinyurl.com/44x2qj4`)

# Concluding...

Enjoy LxMLS 2021!